Lifting Reducible Galois Representations

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29 April 2020

- Let S be a finite set of prime numbers
  - $\mathbb{Q}_S$ is the maximal extension of $\mathbb{Q}$ unramified outside S.
  - Set $\mathrm{G}_S := \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$.
- Let $f \in S_k(\Gamma_1(N))$ be a normalized cuspidal Hecke eigenform and p an odd prime number.
- Fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, the p-adic Galois representation associated to f is denoted by

$$\rho_f : \mathrm{G}_S \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p).$$

- Here, S is the set of primes which divide Np.

The representation ρ_f is a continuous Galois representation which satisfies some characteristic properties, namely,

1. ρ_f is irreducible,
2. ρ_f is unramified outside a finite set of primes,
3. ρ_f is de Rham when restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$,
4. ρ_f is odd, i.e. $\det \rho_f(c) = -1$.

The Fontaine-Mazur Conjecture

- Fontaine and Mazur referred to such continuous Galois representations $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ as Geometric Galois representations.
- These are by definition irreducible, unramified at all but finitely many primes, odd and potentially semistable at p.
- They conjectured that any 2-dimensional Geometric Galois representation ρ arises from a cuspidal Hecke eigenform f, i.e. $\rho \simeq \rho_f$.
- This has been resolved (except for a few cases) following the work of Taylor, Kisin, Emerton and Skinner-Wiles.

Serre's Conjecture

Theorem of Khare and Wintenberger 2008

Let

$$\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$$

be *irreducible*, odd and unramified outside finitely many primes. Then $\bar{\rho}$ lifts to a the Galois representation associated to a cuspidal Hecke eigenform f

$$\begin{array}{ccc} & & \mathrm{GL}_2(\bar{\mathbb{Z}}_p) \\ & \overset{\rho_f}{\nearrow} & \downarrow \\ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & \xrightarrow[\bar{\rho}]{} & \mathrm{GL}_2(\bar{\mathbb{F}}_p). \end{array}$$

Serre's Conjecture for Reducible Representations

Let p be an odd prime and $\mathbb{F}_q$ a finite field of characteristic p and $\mathcal{O} := W(\mathbb{F}_q)$ the ring of Witt-vectors with residue field $\mathbb{F}_q$. This is the valuation ring of the unramified extension of $\mathbb{Q}_p$ of degree $[\mathbb{F}_q : \mathbb{F}_p]$.

Hamblen and Ramakrishna, 2008

Let S be a finite set of primes and

$$\bar{\rho} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : \mathrm{G}_S \to \mathrm{GL}_2(\mathbb{F}_q)$$

1. be reducible, odd and indecomposable,
2. satisfy some favorable conditions.

Then there is a cuspidal Hecke eigenform f such that ρ_f lifts $\bar{\rho}$.

$$
\begin{array}{ccccc}
 & & & & \mathrm{GL}_2(\mathcal{O}) \\
 & & & \overset{\rho_f}{\nearrow} & \downarrow \\
\mathrm{G}_{S\cup X} & \longrightarrow & \mathrm{G}_S & \xrightarrow{\bar{\rho}} & \mathrm{GL}_2(\mathbb{F}_q).
\end{array}
$$

Generalization to Higher Dimensions

Theorem-R

Let

$$\bar{\rho} : \mathsf{G}_S \to \mathsf{GSp}_{2n}(\mathbb{F}_q)$$

1. be reducible, odd and indecomposable,
2. satisfy some natural and mild conditions.

Then there is a finite set of primes X a geometric Galois representation $\rho : \mathsf{G}_{S \cup X} \to \mathsf{GSp}_{2n}(\mathcal{O})$ such that ρ lifts $\bar{\rho}$.

$$
\begin{array}{ccccc}
 & & & & \mathrm{GSp}_{2n}(\mathcal{O}) \\
 & & \overset{\rho}{\nearrow} & & \downarrow \\
\mathrm{G}_{S\cup X} & \longrightarrow & \mathrm{G}_S & \xrightarrow{\bar{\rho}} & \mathrm{GSp}_{2n}(\mathbb{F}_q).
\end{array}
$$

- The case when $\bar{\rho} : \mathrm{G}_S \to \mathrm{GL}_2(\mathbb{F}_q)$ is irreducible goes back to Ramakrishna (2002). This was proved before Serre's conjecture was proved by Khare and Wintenberger (2009).
- Patrikis (2006) generalized Ramakrishna's theorem to higher dimensional irreducible representations $\bar{\rho} : \mathrm{G}_S \to \mathrm{GSp}_{2n}(\mathbb{F}_q)$ for $n \geq 2$.

Galois Deformations

- The category of coefficient rings $\mathcal{C}_{\mathcal{O}}$ is comprised of complete Noetherian local rings R with maximal ideal $\mathfrak{m}_R$ such that the residue-field $R/\mathfrak{m}_R \simeq \mathbb{F}_q$. Further, R is endowed with the structure of an $\mathcal{O}$-algebra.
- A coefficient ring R has a presentation

$$R \simeq \mathcal{O}[[X_1, \ldots, X_k]]/(f_1, \ldots, f_r).$$

- Two lifts $\rho_1, \rho_2 : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}_{2n}(R)$ of $\bar{\rho}$ are strictly equivalent if for some $A \in \mathrm{GSp}_{2n}(R)$ such that $A \equiv \mathrm{Id} \mod \mathfrak{m}_R$,

$$\rho_1 = A\rho_2 A^{-1}.$$

- An equivalence class of lifts is a Galois deformation.

- The strategy involves lifting step by step by producing a compatible collection of deformations $\rho_m : \mathrm{G}_{S \cup X} \to \mathrm{GSp}_{2n}(\mathcal{O}/p^m)$ for $m \geq 1$.
- Fix a lift $\psi : \mathrm{G}_S \to \mathcal{O}^\times$ of the similitude character of $\bar{\rho}$. We shall consider deformations with fixed similitude character ψ.
- $\mathrm{Ad}^0 \bar{\rho}$ will denote the Lie algebra of sp_{2n} over $\mathbb{F}_q$ equipped with Galois action defined by

$$g \cdot v = \bar{\rho}(g) v \bar{\rho}(g)^{-1}.$$

- One may identify $\mathrm{Ad}^0 \bar{\rho}$ with the kernel of the reduction map

$$\mathrm{Sp}_{2n}(\mathcal{O}/p^{m+1}) \to \mathrm{Sp}_{2n}(\mathcal{O}/p^m)$$

by identifying $v \in \mathrm{sp}_{2n}(\mathbb{F}_q)$ with $\mathrm{Id} + p^m v$.

- There is a cohomological obstruction to lifting ρ_m to a representation to $\mathrm{GSp}_{2n}(\mathcal{O}/p^{m+1})$

$$\rho_{m+1} : \mathrm{G}_{S \cup X} \to \mathrm{GSp}_{2n}(\mathcal{O}/p^{m+1}).$$

- This obstruction is a class

$$\mathcal{O}(\rho_m) \in H^2(\mathrm{G}_{S \cup X}, \mathrm{Ad}^0 \bar{\rho}).$$

- Suppose there is a lift ρ_m of $\bar{\rho}$ and that the obstruction-class $\mathcal{O}(\rho_m) = 0$. This means that there is a lift ρ_{m+1} of ρ_m.

- The set of lifts $\rho_{m+1} : \mathrm{G}_{S\cup X} \to \mathrm{GSp}_{2n}(\mathcal{O}/p^{m+1})$ of ρ_m is an $H^1(\mathrm{G}_{S\cup X}, \mathrm{Ad}^0\bar{\rho})$-pseudotorsor. This means that if ρ_{m+1} and ρ'_{m+1} are two lifts of ρ_m, then there is a unique cohomology class $f \in H^1(\mathrm{G}_{S\cup X}, \mathrm{Ad}^0\bar{\rho})$ such that

$$\rho'_{m+1} = (\mathrm{Id} + p^m f)\rho_{m+1}.$$

- Suppose ρ_m lifts to ρ_{m+1}. Then, it is shown that there is a cohomology class f such that after replacing ρ_{m+1} by the twist $(\mathrm{Id} + p^m f)\rho_{m+1}$, the obstruction

$$\mathcal{O}(\rho_{m+1}) = 0.$$

  This means that ρ_{m+1} lifts to ρ_{m+2}, at which stage the inductive argument is repeated.

Local Deformations

- It is not very clear at the outset how one may choose a class f at each step which does the job. One notices that the global deformations ρ_m in question satisfy various local deformation conditions.
- Let v be a prime at which ρ_m is allowed to ramify. Fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_v$ and a compatible inclusion of $\mathrm{G}_{\mathbb{Q}_v} \hookrightarrow \mathrm{G}_{\mathbb{Q}}$.
- Let $\bar{\rho}_{\restriction v}$ denote the restriction of $\bar{\rho}$ to $\mathrm{G}_{\mathbb{Q}_v}$.

- Let
$$\mathrm{Def}_v : \mathcal{C}_{\mathcal{O}} \to \mathrm{Sets}$$
be the functor of deformations of $\bar{\rho}_{\restriction v}$. It is representable by a scheme $\mathrm{Spec}\,\mathcal{R}_v$, where $\mathcal{R}_v \in \mathcal{C}_{\mathcal{O}}$ with maximal ideal $\mathfrak{m}_v$.
- The tangent space
$$(\mathfrak{m}_v/\mathfrak{m}_v^2)^* \simeq H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0\,\bar{\rho}).$$
This tangent space is also in canonical bijection with the infinitesimal deformations to the dual numbers $\mathrm{Def}_v(\mathbb{F}[\epsilon]/(\epsilon^2))$.

At each prime $v \in S$ there is a natural subfunctor $\mathcal{C}_v$ of Def_v, which is represented by a closed subscheme

$$\mathrm{Spec}\, R_v \hookrightarrow \mathrm{Spec}\, \mathcal{R}_v$$

satisfying a number of special properties.

1. The subscheme $\mathrm{Spec}\, R_v$ is smooth.
2. The tangent space of $\mathrm{Spec}\, R_v$ is denoted

$$\mathcal{N}_v \subseteq H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0\, \bar{\rho}).$$

   The dimension of $\mathcal{N}_v$ is equal to $\dim H^0(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0\, \bar{\rho}) + \delta_v$, where δ_v is 1 if $v = p$ and 0 otherwise.

Selmer and dual-Selmer conditions

- At each prime $v \in S$, the space $\mathcal{N}_v \subseteq H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho})$.
- Set $\mathrm{Ad}^0 \bar{\rho}^* := \mathrm{Hom}(\mathrm{Ad}^0 \bar{\rho}, \mu_p)$, there is a non-degenerate pairing

$$H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho}) \times H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho}^*) \to \mathbb{F}.$$

- The orthogonal-complement to $\mathcal{N}_v$ is denoted by $\mathcal{N}_v^{\perp} \subseteq H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho}^*)$.

- The Selmer group attached to the Selmer-data $\mathcal{N} := \{\mathcal{N}_v\}_{v \in S}$ is the kernel to the restriction-map

$$H^1_{\mathcal{N}}(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho})$$

$$:= \ker\{H^1(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}) \to \prod_{v \in S} H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho})/\mathcal{N}_v\}$$

  consists of cohomology classes which are unramified outside S and lie in $\mathcal{N}_v$ at each prime $v \in S$.
- The dual Selmer group is

$$H^1_{\mathcal{N}^\perp}(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}^*)$$

$$:= \ker\{H^1(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}^*) \to \prod_{v \in S} H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho}^*)/\mathcal{N}_v^\perp\}.$$

- A formula of Wiles expresses the difference between the dimensions of the Selmer and dual-Selmer group in terms of quantities associated to the local tangent spaces $\mathcal{N}_v$. More precisely,

$$h^1_{\mathcal{N}}(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}) - h^1_{\mathcal{N}^\perp}(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}^*)$$
$$= h^0(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}) - h^0(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}^*)$$
$$+ \sum_{v \in S \cup \{\infty\}} \left(\dim \mathcal{N}_v - h^0(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho})\right).$$

- On applying this formula, we have that

$$\delta := h^1_{\mathcal{N}}(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}) = h^1_{\mathcal{N}^\perp}(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}^*).$$

A special case

- Let us consider the special case when $\delta = 0$, which is to say that
$$H^1_{\mathcal{N}}(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}) = 0$$
and
$$H^1_{\mathcal{N}^\perp}(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}^*) = 0$$
.
- There is a long-exact sequence in cohomology attributed to Poitou-Tate. Let Φ_S denote the restriction map
$$\Phi_S : H^1(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}) \to \prod_{v \in S} H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho})/\mathcal{N}_v.$$

- The kernel of Φ_S is $H^1_{\mathcal{N}}(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho})$ by definition.
- By Poitou-Tate, the cokernel of Φ_S is contained in $H^1_{\mathcal{N}^\perp}(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}^*)$.
- Therefore, when $\delta = 0$, the map

$$\Phi_S : H^1(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}) \to \prod_{v \in S} H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho})/\mathcal{N}_v$$

  is an isomorphism.
- Let $Ш^2_S(\mathrm{Ad}^0 \bar{\rho})$ denote the kernel of the restriction map

$$H^2(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}) \to \prod_{v \in S} H^2(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho}).$$

- It is a consequence of global-duality that $\text{Ш}^2_S(\mathrm{Ad}^0\,\bar{\rho})^\vee$ is contained in $H^1_{\mathcal{N}^\perp}(\mathrm{G}_S, \mathrm{Ad}^0\,\bar{\rho}^*)$, hence, $\text{Ш}^2_S(\mathrm{Ad}^0\,\bar{\rho}) = 0$.
- Suppose that $\rho_m : \mathrm{G}_S \to \mathrm{GSp}_{2n}(\mathcal{O}/p^m)$ is a lift of $\bar{\rho}$ which satisfies the conditions $\mathcal{C}_v$ at the primes $v \in S$.
- Since $\mathcal{C}_v$ is smooth, the restriction $\rho_{m\restriction v}$ lifts to a representation

$$\mathrm{G}_{\mathbb{Q}_v} \to \mathrm{GSp}_{2n}(\mathcal{O}/p^{m+1}).$$

- Therefore, the obstruction to lifting ρ_n is locally trivial at the primes $v \in S$. In other words,

$$\mathcal{O}(\rho_n) \in \text{Ш}^2_S(\operatorname{Ad}^0 \bar{\rho}).$$

  Since $\text{Ш}^2_S(\operatorname{Ad}^0 \bar{\rho}) = 0$, it follows that ρ_m must lift to ρ_{m+1} if ρ_m satisfies the conditions $\mathcal{C}_v$ at each prime $v \in S$.
- To continue the inductive argument it suffices to show that on replacing ρ_{m+1} by a twist

$$(\operatorname{Id} + p^m f)\rho_{m+1},$$

  it may be assumed that ρ_{m+1} also satisfies $\mathcal{C}_v$ at each prime $v \in S$.

- For $v \in S$, there is a choice of $f_v \in H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho})$ such that

$$(\mathrm{Id} + p^m f_v)\rho_{m+1\restriction v} \in \mathcal{C}_v.$$

  The choice of f_v is unique up to $\mathcal{N}_v$.
- Since the localization map

$$\Phi_S : H^1(\mathrm{G}_S, \mathrm{Ad}^0 \bar{\rho}) \to \prod_{v \in S} H^1(\mathrm{G}_{\mathbb{Q}_v}, \mathrm{Ad}^0 \bar{\rho})/\mathcal{N}_v$$

  is an isomorphism, there is a global cohomology class f such that $\Phi_S(f) = (f_v)_{v \in S}$.

The general case

- In the general case, we may adjoin a set of $\leq \delta$ primes X to S so that the Selmer and dual Selmer groups

$$H^1_{\mathcal{N}}(\mathrm{G}_{S \cup X}, \mathrm{Ad}^0 \bar{\rho}) = 0$$

and

$$H^1_{\mathcal{N}^\perp}(\mathrm{G}_{S \cup X}, \mathrm{Ad}^0 \bar{\rho}^*) = 0.$$

- These auxiliary primes $v \in X$ are called *trivial* primes and are each equipped with a deformation problem $\mathcal{C}_v$ and versal tangent space $\mathcal{N}_v$.

Trivial primes

- A prime $v \notin S$ is said to be trivial if $G_{\mathbb{Q}_v} \subset \ker \bar{\rho}$, i.e the restriction $\bar{\rho}_{\restriction v}$ is trivial. Further, $v \equiv 1 \mod p$ and $v \not\equiv 1 \mod p^2$.
- There is a deformation problem $\mathcal{C}_v$, which we describe in the GL_2 case. Let σ_v be a Frobenius at v and τ_v a generator of pro-p tame-intertia. They satisfy a relation

$$\sigma_v \tau_v \sigma_v^{-1} = \tau_v^v.$$

- Deformations $\varrho \in \mathcal{C}_v$ are tamely ramified for which

$$\varrho(\sigma_v) = c \cdot \begin{pmatrix} v & x \\ 0 & 1 \end{pmatrix}$$

  and

$$\varrho(\tau_v) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

  Here, x, y are arbitrary and c is determined since the determinant of ϱ is fixed.
- The deformation problem $\mathcal{C}_v$ is not representable. There is however a 3-dimensional space $\mathcal{N}_v$ which plays the role of a versal tangent space.

- The space stablizes $\mathcal{C}_v(\mathcal{O}/p^N)$ for $N \geq 3$, but not $\mathcal{C}_v(\mathcal{O}/p^2)$.
- In greater detail, if $\varrho \in \mathcal{C}_v(\mathcal{O}/p^N)$ for $N \geq 3$ and $f \in \mathcal{N}_v$ then the twist
$$(\mathrm{Id} + p^{N-1} f)\varrho \in \mathcal{C}_v(\mathcal{O}/p^N).$$

Lifting to characteristic-zero

It is shown that there is a set of $\leq \delta + 2$ of trivial primes X such that

- X kills the dual-Selmer group

$$H^1_{\mathcal{N}^\perp}(\mathrm{G}_{S\cup X}, \mathrm{Ad}^0\,\bar{\rho}^*) = 0,$$

- there exists a lift $\rho_3 : \mathrm{G}_{S\cup X} \to \mathrm{GSp}_{2n}(\mathcal{O}/p^3)$ of $\bar{\rho}$ which satisfies the conditions $\mathcal{C}_v$ at all primes $v \in S$. Further, the lift ρ_3 is shown to be *irreducible*.

The arguments explained in the special case when $\delta = 0$ generalize to this setting to give a characteristic zero lift $\rho : \mathrm{G}_{S\cup X} \to \mathrm{GSp}_{2n}(\mathcal{O})$ which satisfies the conditions $\mathcal{C}_v$ for $v \in S \cup X$ and is irreducible.

Deformation functors with fixed local constraints

- Let X be a set of trivial primes disjoint from S and $\mathcal{F}$ be the functor of deformations of $\bar{\rho}$ which are unramified outside $S \cup X$ and satisfy the conditions $\mathcal{C}_v$ at the primes $v \in S \cup X$. The functor $\mathcal{F}$ need not be representable, since the deformation problem $\mathcal{C}_v$ at each trivial prime $v \in X$ is not representable.
- We compare this setting with the case when $\bar{\rho}$ is irreducible. In this setting, the set X consists of *nice*-primes. At each nice prime there is a deformation condition which is representable. The functor of deformations unramified outside $S \cup X$ satisfying the local conditions $\mathcal{C}_v$ for $v \in S \cup X$ is representable.

- This is to say that there a universal deformation $\rho_{\mathcal{F}} : \mathsf{G}_{S \cup X} \to \mathsf{GSp}_{2n}(\mathcal{R}_{\mathcal{F}})$, where $\mathcal{R}_{\mathcal{F}} \in \mathcal{C}_{\mathcal{O}}$ is the universal deformation ring w.r.t the conditions $\{\mathcal{C}_v\}_{v \in S \cup X}$.
- When the determinant is fixed and

$$H^1_{\mathcal{N}^\perp}(\mathsf{G}_{S \cup X}, \mathrm{Ad}^0 \bar{\rho}) = 0,$$

  the universal deformation ring is $\mathcal{O}$.
- When the determinant is not fixed and X kills the appropriate dual Selmer group, the universal deformation ring is isomorphic to $\mathcal{O}[[X]]$.

- We return to the case when $\bar{\rho}$ is a reducible 2-dimensional representation. Let X be a collection of trivial primes which kill the Selmer group and such that there is a characteristic zero lift

$$\rho : \mathrm{G}_{S \cup X} \to \mathrm{GL}_2(\mathcal{O}).$$

A Modified Deformation Functor

- Let $\mathfrak{C}$ be the subcategory of $\mathcal{C}_{\mathcal{O}}$ consisting of R such that $p \notin \mathfrak{m}_R^2$.
- Let $\rho_2 := \rho \mod p^2$, set

$$\mathrm{Def} : \mathfrak{C} \to \mathrm{Sets}$$

be such that $\mathrm{Def}(R)$ consists of deformations

$$\varrho : \mathrm{G}_{S \cup X} \to \mathrm{GL}_2(R)$$

of $\bar{\rho}$ for which

1. ϱ is unramified outside $S \cup X$ and satisfies the conditions $\mathcal{C}_v$ at the primes $v \in S \cup X$.
2. $\varrho \mod m_R^2$ coincides with $\rho_2 : \mathrm{G}_{S \cup X} \to \mathrm{GL}_2(\mathcal{O}/p^2) \to \mathrm{GL}_2(R/\mathfrak{m}_R^2)$.

Theorem- R

Let $\bar{\rho} : \mathrm{G}_S \to \mathrm{GL}_2(\mathbb{F}_q)$ be a reducible Galois representation. Assume that $\bar{\rho}$ satisfies a number of favorable conditions. There is an irreducible, odd and p-ordinary deformation

$$\tilde{\rho} : \mathrm{G}_{S \cup X} \to \mathrm{GL}_2(\mathcal{O}[[X]])$$

which is a hull for Def. In other words, if $R \in \mathfrak{C}$ and $\rho' : \mathrm{G}_{S \cup X} \to \mathrm{GL}_2(R)$ satisfies Def, then there is a morphism $\phi : \mathcal{O}[[X]] \to R$ which induces ρ' via $\tilde{\rho}$.

$$\begin{array}{ccc} & & \mathrm{GL}_2(\mathcal{O}[[X]]) \\ & \overset{\tilde{\rho}}{\nearrow} & \downarrow \\ \mathrm{G}_{S \cup X} & \xrightarrow[\rho']{} & \mathrm{GL}_2(R). \end{array}$$

- The map ϕ is not necessarily unique.
- There is a natural decomposition $\mathcal{G} := \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \simeq \Delta \times \Gamma$, where $\Delta \simeq (\mathbb{Z}/p)^\times$ and $\Gamma \simeq \mathbb{Z}_p$. The Iwasawa algebra $\Lambda_{\mathcal{O}}$ is the pro-group ring

$$\Lambda_{\mathcal{O}} := \varprojlim_{U} \mathcal{O}[\Gamma/U]$$

  where U ranges over all finite index subgroups of Γ.
- On setting $\gamma - 1 = T$, there is an isomorphism $\Lambda_{\mathcal{O}} \simeq \mathcal{O}[[T]]$.

The Map to Weight-Space

- Let $\gamma \in \Gamma$ be a choice of topological generator of Γ and R a coefficient ring. Let

$$\rho' : \mathrm{G}_{S \cup X} \to \mathrm{GL}_2(R)$$

be a deformation of $\bar{\rho}$, the *weight* of ρ' is the point

$$\mathcal{W}t(\rho') : \operatorname{Spec} R \to \operatorname{Spec} \Lambda_{\mathcal{O}}$$

induced by the homomorphism of rings mapping T to $\det \rho'(\gamma) - 1$.

- Let R be a coefficient ring in $\mathfrak{C}$ with maximal ideal $\mathfrak{m}_R$. The functor of points of the map to weight space induces on R-points a map

$$\mathcal{W}t^* : \mathrm{Def}(R) \to \mathrm{Hom}_{\mathfrak{C}}(\Lambda_{\mathcal{O}}, R)$$

  whose image consists weights in the mod $(pR) \cap \mathfrak{m}_R^2$ congruence class of weights which coincide with the weight of the chosen lift ρ_2 modulo $(pR) \cap \mathfrak{m}_R^2$.
- When $R = \mathcal{O}$, we have that $pR \cap \mathfrak{m}_R^2 = p^2\mathcal{O}$. Therefore, $\tilde{\rho}$ interpolates a mod p^2 congruence class of weights for various eigencuspforms lifting $\bar{\rho}$.

Constructing Certain Special Galois Extensions

- Let $p > 2$ be a prime number and K a number field. Let L be a Galois extension of K such that $\mathrm{Gal}(L/K)$ is isomorphic to a p-adic Lie-group. Suppose that L/K is infinite.
- The tame Fontaine-Mazur conjecture posits that there is no such extension L of a number field K such that
  1. finitely many primes of K ramify in L,
  2. all primes of K above p are finitely ramified in L.
- It is natural to ask if such extensions exist once we pass up $\mathbb{Q}(\mu_{p^\infty})$.

- Ohtani and Blondeau independently constructed special Galois extensions of $\mathbb{Q}(\mu_{p^\infty})$ which had surprisingly little ramification.
- Ohtani's results were modified by Blondeau, who showed that for certain primes p, there exists a Galois extension $L/\mathbb{Q}(\mu_{p^\infty})$ for which
  1. L is unramified at the prime of $\mathbb{Q}(\mu_{p^\infty})$ above p and ramified at finitely many primes,
  2. $\mathrm{Gal}(L/\mathbb{Q}(\mu_{p^\infty})) \simeq \mathrm{SL}_2(\mathbb{Z}_p)$.
- Blondeau proves his result by lifting certain irreducible Galois representations

$$\bar{\rho} : \mathrm{G}_S \to \mathrm{GL}_2(\mathbb{F}_p)$$

for which

$$\bar{\rho}_{\restriction \mathrm{G}_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}.$$

- It is shown that $\bar{\rho}$ lifts to $\rho : \mathrm{G}_{S\cup X} \to \mathrm{GL}_2(\mathbb{Z}_p)$ such that

$$\rho_{\restriction \mathrm{G}_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \tilde{\varphi}_1 & 0 \\ 0 & \tilde{\varphi}_2 \end{pmatrix}.$$

- Further, the restriction $\rho_{\restriction \mathbb{Q}(\mu_{p^\infty})} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_{p^\infty})) \to \mathrm{SL}_2(\mathbb{Z}_p)$ is arranged to be surjective.
- Let $L/\mathbb{Q}(\mu_{p^\infty})$ be the fixed field of the kernel of this homomorphism, the Galois group

$$\mathrm{Gal}(L/\mathbb{Q}(\mu_{p^\infty})) \simeq \mathrm{SL}_2(\mathbb{Z}_p)$$

and is unramified at the prime above p.

- The drawback is that there are very few examples of irreducible Galois representations $\bar{\rho} : \mathrm{G}_S \to \mathrm{GL}_2(\mathbb{F}_p)$ which decompose into a direct sum of characters when restricted to $\mathrm{G}_{\mathbb{Q}_p}$

$$\bar{\rho}_{\restriction \mathrm{G}_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}.$$

- In fact, there are four primes p at which Blondeau constructs examples of such $\bar{\rho}$, namely, $p = 107, 139, 271$ and 379.
- However, there are many examples of reducible Galois representations which decompose when restricted to $\mathrm{G}_{\mathbb{Q}_p}$.

- We associate a Galois representation to class group data. Denote by
$$\mathcal{C} := \mathrm{Cl}(\mathbb{Q}(\mu_p)) \otimes \mathbb{F}_p$$
and $\bar{\chi}$ the mod p cyclotomic character. The Galois module $\mathcal{C}$ decomposes into isotypic components
$$\mathcal{C} = \bigoplus_{i=0}^{p-2} \mathcal{C}(\bar{\chi}^i).$$
- An $\mathbb{F}_p$-quotient of $\mathcal{C}(\bar{\chi}^i)$ coincides via inflation-restriction with a class $\beta \in H^1(\mathrm{G}_{\{p\}}, \mathbb{F}_p(\bar{\chi}^i))$. This class in turn defines a reducible Galois representation
$$\bar{\rho} = \begin{pmatrix} \bar{\chi}^i & \beta \\ 0 & 1 \end{pmatrix} : \mathrm{G}_{\{p\}} \to \mathrm{GL}_2(\mathbb{F}_p).$$

- It is not hard to show that $\bar{\rho}_{\restriction G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \bar{\chi}^i & 0 \\ 0 & 1 \end{pmatrix}$.
- We show that one may lift $\bar{\rho}$ to a characteristic zero Galois representation

$$\rho : G_{\{p\}\cup X} \to GL_2(\mathbb{Z}_p)$$

  for which $\rho_{\restriction G_{\mathbb{Q}_p}}$ decomposes into a product of characters.
- The image of the restriction $\rho_{\restriction \mathbb{Q}(\mu_{p^\infty})} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_{p^\infty})) \to SL_2(\mathbb{Z}_p)$ contains the principal congruence subgroup.
- Let $L/\mathbb{Q}(\mu_{p^\infty})$ be the fixed field of the kernel of this homomorphism, the Galois group

$$\mathrm{Gal}(L/\mathbb{Q}(\mu_{p^\infty})) \subset SL_2(\mathbb{Z}_p)$$

  contains the principal congruence subgroup. Further, it is unramified at the prime above p.

Theorem- R

Suppose that $p \geq 5$ be a prime and $i \neq \frac{p-1}{2}$ an odd integer between $2 \leq i \leq p-3$ for which the isotypic space $\mathcal{C}(\bar{\chi}^i) \neq 0$. There are infinitely many Galois extensions $L/\mathbb{Q}(\mu_{p^\infty})$ for which

- the Galois group $\mathrm{Gal}(L/\mathbb{Q}(\mu_{p^\infty}))$ topologically isomorphic to a subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ which contains the principal congruence subgroup.
- F is unramified at the prime above p and ramified above finitely many rational primes.

Thank you!